Mathematical behavior of solutions for a logarithmic p-Laplacian equation with distributed delay

Erhan Pişkin[1] and Hazal Yüksekkaya[2]

[1,2]Dicle University, Department of Mathematics, Diyarbakir, Turkey
E-mail: episkin@dicle.edu.tr[1], hazally.kaya@gmail.com[2]

Abstract

In this article, we concerned with a logarithmic p-Laplacian equation with distributed internal delay. Firstly, we obtain the global existence of solutions by utilizing the well-depth method. Later, under appropriate assumptions on the weight of the delay and that of frictional damping, we establish the exponential decay. Moreover, we obtain the blow up results for negative initial energy.

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1 Introduction

In this work, we deal with the logarithmic p-Laplacian equation with distributed delay as follows:

$$\begin{cases} u_{tt} - \Delta u - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) + \mu_1 u_t(x,t) + \int_{\tau_1}^{\tau_2} \mu_2(s) u_t(x,t-s)\, ds & \\ = u|u|^{q-2} \ln |u|^k, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \\ u_t(x,-t) = f_0(x,t), & \text{in } (0,\tau_2), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of R^n $(n \geq 1)$ with a smooth boundary $\partial\Omega$. k, μ_1 are positive constants, the term $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$ is called p-Laplacian, the integral term denotes the distributed delay for $0 \leq \tau_1 < \tau_2$ and $\mu_2 : [\tau_1, \tau_2] \to [0,\infty)$ is a bounded function. u_0, u_1, f_0 are the initial data functions to be specified later.

Problems about the mathematical behavior of solutions for PDEs with time delay effects have become interesting for many authors mainly because time delays often appear in many practical problems such as thermal, economic phenomena, physical, chemical, biological, mechanical applications, electrical engineering systems and medicine [12]. Generally, logarithmic nonlinearity appears in nuclear physics, inflation cosmology, geophysics and optics (see [3, 8]).

Firstly, for the literature review, we begin with the works of Birula and Mycielski [4, 5]. They investigated the following equation with logarithmic term:

$$u_{tt} - u_{xx} + u - \varepsilon u \ln |u|^2 = 0. \tag{1.2}$$

They are the pioneer of these kind of problems. They established that, in any number of dimensions, wave equations including the logarithmic term have localized, stable, soliton-like solutions.

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In 1980, Cazenave and Haraux [6] concerned with the following equation:

$$u_{tt} - \Delta u = u \ln |u|^k . \tag{1.3}$$

The authors obtained the existence and uniqueness of the solutions of the equation (1.3). Gorka [8] obtained the global existence for one-dimensional of the equation (1.3). Bartkowski and Gorka [3], studied the weak solutions and established the existence results.

In 1986, Datko et al. [7] showed that, a small delay effect is a source of instability. In [14], Nicaise and Pignotti studied the following equation:

$$u_{tt} - \Delta u + a_0 u_t (x,t) + a u_t (x, t-\tau) = 0, \tag{1.4}$$

where a_0, $a > 0$. The authors established that, under the condition $0 \leq a \leq a_0$, the system is exponentially stable. They proved a sequence of delays that shows the solution is instable in the case $a \geq a_0$.

In [15], Nicaise and Pignotti introduced the distributed delay:

$$\int_{\tau_1}^{\tau_2} \mu_2 (s) u_t (x, t-s) \, ds. \tag{1.5}$$

Under appropriate conditions, they established the exponential stability results of the wave equation with boundary or internal distributed delay.

Messaoudi and Kafini [11], studied the wave equation with delay as follows:

$$u_{tt} - \operatorname{div} \left(|\nabla u|^{m-2} \nabla u \right) + \mu_1 u_t (x,t) + \mu_2 u_t (x, t-\tau) = b |u|^{p-2} u. \tag{1.6}$$

Under suitable conditions, they proved the global nonexistence of the equation (1.6).

Nhan and Truong [13], concerned with the following equation with logarithmic term:

$$u_t - \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) - \Delta u_t = |u|^{p-2} u \ln |u| . \tag{1.7}$$

They proved existence, decay and blow up results for the equation (1.7).

In [10], Kafini and Messaoudi studied the following wave equation with delay and logarithmic terms:

$$u_{tt} - \Delta u + \mu_1 u_t (x,t) + \mu_2 u_t (x, t-\tau) = |u|^{p-2} u \log |u|^k . \tag{1.8}$$

They established the local existence and blow up results of the equation (1.8).

In the absence of the p-Laplacian term ($\operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right)$), in [9], Kafini studied the following wave equation:

$$u_{tt} - \Delta u + \mu_1 u_t (x,t) + \int_{\tau_1}^{\tau_2} \mu_2 (s) u_t (x, t-s) = u |u|^{p-2} \ln |u|^k , \tag{1.9}$$

the author established the local and global existence. Moreover, he proved the exponential decay of solutions for the equation (1.9). Recently, some other authors studied hyperbolic type equations (see [2, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]).

In this paper, we study the global existence, exponential decay and blow up of solutions for the logarithmic p-Laplacian equation (1.1) with distributed delay, motivated by above works. To our best knowledge, there is no research, related to the logarithmic p-Laplacian equation (1.1) with distributed delay term ($\int_{\tau_1}^{\tau_2} \mu_2(s) u_t(x,t-s)\,ds$) and logarithmic ($u|u|^{q-2}\ln|u|^k$) source term, therefore, our paper is the generalization of the previous studies.

This work consists of five sections in addition to the introduction: Firstly, in Section 2, we give some needed materials. Then, in Section 3, we get the global existence results by the well-depth method. Moreover, in Section 4, we prove the exponential decay of solutions. Finally, in Section 5, we establish the blow up results for negative initial energy.

2 Preliminaries

In this section, we give some materials for our main result. As usual, the notation $\|.\|_p$ denotes L^p norm, and $(.,.)$ is the L^2 inner product. In particular, we write $\|.\|$ instead of $\|.\|_2$.

Let $B_p > 0$ be the constant satisfying [1]

$$\|v\|_2 \leq B_q \|\nabla v\|_q, \text{ for } v \in H_0^1(\Omega). \tag{2.1}$$

Similar to the [14], we introduce the new variable

$$z(x,\rho,s,t) = u_t(x,t-\rho s) \text{ in } \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty).$$

Therefore, we have

$$sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0 \text{ in } \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty).$$

Hence, the problem (1.1) is equivalent to:

$$\begin{cases} u_{tt} - \Delta u - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \mu_1 u_t(x,t) & \\ + \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)\,ds = u|u|^{q-2}\ln|u|^k & \text{in } \Omega \times (\tau_1,\tau_2) \times (0,\infty) \\ sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0 & \text{in } \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty) \\ z(x,\rho,s,0) = f_0(x,-\rho s) & \text{in } \Omega \times (0,1) \times (\tau_1,\tau_2) \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega. \end{cases} \tag{2.2}$$

The energy functional related to the problem (2.2) is, for $\forall t \geq 0$,

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{p}\|\nabla u\|_p^p + \frac{k}{q^2}\|u\|_q^q \\ &\quad + \frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s(\xi + \mu_2(s))|z(x,\rho,s,t)|^2\,ds\,d\rho\,dx \\ &\quad - \frac{1}{q}\int_\Omega |u|^q \ln|u|^k\,dx, \end{aligned} \tag{2.3}$$

where ξ is a positive constants satisfying

$$\mu_1 > \int_{\tau_1}^{\tau_2} \mu_2(s)\,ds + \frac{\xi}{2}(\tau_2 - \tau_1), \tag{2.4}$$

under the condition

$$\mu_1 > \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds.$$

The following lemma shows that the related energy functional of the problem is nonincreasing:

Lemma 2.1. Suppose that (2.4) holds. Then, along the solution of (2.2) and for some $C_0 \geq 0$, we get

$$E'(t) \leq -C_0 \int_\Omega \left(|u_t|^2 + |z(x,1,s,t)|^2 \right) dx \leq 0. \tag{2.5}$$

Proof. By multiplying the first equation in (2.2) by u_t and integrating over Ω and the second equation in (2.2) by $(\xi + \mu_2(s))\, z$ and integrating over $(\tau_1, \tau_2) \times (0,1) \times \Omega$ with respect to s, ρ and x, summing up, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\begin{array}{c} \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{p}\|\nabla u\|_p^p + \frac{k}{q^2}\|u\|_q^q - \frac{1}{q}\int_\Omega |u|^q \ln |u|^k\, dx \\ +\frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s(\xi + \mu_2(s))\, |z(x,\rho,s,t)|^2\, ds\, d\rho\, dx \end{array} \right) \\
= & -\mu_1 \int_\Omega |u_t|^2\, dx - \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} (\xi + \mu_2(s))\, z z_\rho(x,\rho,s,t)\, ds\, d\rho\, dx \\
& - \int_\Omega u_t \int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t)\, ds\, dx.
\end{aligned} \tag{2.6}$$

Now, we handle the last two terms of the right-hand side of (2.6) as:

$$\begin{aligned}
& -\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} (\xi + \mu_2(s))\, z z_\rho(x,\rho,s,t)\, ds\, d\rho\, dx \\
= & -\frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} \int_0^1 \frac{\partial}{\partial \rho}\left[(\xi + \mu_2(s))\, |z(x,\rho,s,t)|^2 \right] d\rho\, ds\, dx \\
= & \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} \mu_2(s)\, ds + \xi(\tau_2 - \tau_1) \right) \int_\Omega |u_t|^2\, dx \\
& -\frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} (\xi + \mu_2(s))\, |z(x,1,s,t)|^2\, ds\, dx
\end{aligned}$$

and

$$\begin{aligned}
& -\int_\Omega u_t \int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t)\, ds\, dx \\
\leq & \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} \mu_2(s)\, ds \int_\Omega |u_t|^2\, dx + \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds \int_\Omega |z(x,1,s,t)|^2\, dx \right).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\frac{dE(t)}{dt} \leq & -\left(\mu_1 - \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds - \frac{\xi}{2}(\tau_2 - \tau_1) \right) \int_\Omega |u_t|^2\, dx \\
& -\frac{\xi}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |z(x,1,s,t)|^2\, ds\, dx.
\end{aligned}$$

By using (2.4), we obtain, for some $C_0 > 0$,

$$E'(t) \leq -C_0 \int_{\Omega} \left(|u_t|^2 + \int_{\tau_1}^{\tau_2} |z(x,1,s,t)|^2 \, ds \right) dx \leq 0.$$

Q.E.D.

3 Global existence

In this section, we establish that the solution of (2.2) is uniformly bounded and global in time. For this aim, we set

$$I(t) = \|\nabla u\|^2 + \|\nabla u\|_p^p - \int_{\Omega} |u|^q \ln |u|^k \, dx,$$

$$\begin{aligned} J(t) &= \frac{1}{2} \|\nabla u\|^2 + \frac{1}{p} \|\nabla u\|_p^p + \frac{k}{q^2} \|u\|_q^q \\ &+ \frac{1}{2} \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} s(\xi + \mu_2(s)) z^2 ds d\rho dx - \frac{1}{q} \int_{\Omega} |u|^q \ln |u|^k \, dx. \end{aligned} \tag{3.1}$$

Therefore,

$$E(t) = J(t) + \frac{1}{2} \|u_t\|^2.$$

Lemma 3.1. Assume that the initial data $u_0,\ u_1 \in H_0^1(\Omega) \times L^2(\Omega)$ satisfying

$$I(0) > 0 \text{ and } \beta = \min \left\{ kC_{q+l} \left(\frac{2qE(0)}{q-2} \right)^{\frac{q-2+l}{2}},\ kC_{q+l} \left(\frac{pq}{q-p} E(0) \right)^{\frac{q-p+l}{p}} \right\} < 1. \tag{3.2}$$

Then, $I(t) > 0$, for any $t \in [0, T]$.

Proof. Since $I(0) > 0$ we infer by continuity that there exists $T^* \leq T$ such that $I(t) \geq 0$ for all $t \in [0, T^*]$. This implies that, for all $t \in [0, T^*]$,

$$\begin{aligned} J(t) &= \frac{q-2}{2q} \|\nabla u\|^2 + \frac{q-p}{pq} \|\nabla u\|_p^p + \frac{k}{q^2} \|u\|_q^q \\ &+ \frac{1}{2} \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} s(\xi + \mu_2(s)) z^2 ds d\rho dx + \frac{1}{q} I(t). \end{aligned}$$

$$J(t) \geq \frac{q-2}{2q} \|\nabla u\|^2 + \frac{q-p}{pq} \|\nabla u\|_p^p.$$

Hence,

$$\|\nabla u\|^2 \leq \frac{2q}{q-2} J(t) \leq \frac{2q}{q-2} E(t) \leq \frac{2q}{q-2} E(0), \tag{3.3}$$

and

$$\|\nabla u\|_p^p \leq \frac{pq}{q-p} J(t) \leq \frac{pq}{q-p} E(t) \leq \frac{pq}{q-p} E(0). \tag{3.4}$$

By using the fact that $\ln|u| < |u|^l$, we get

$$\int_\Omega |u|^q \ln|u|\, dx \le \int_\Omega |u|^{q+l}\, dx, \tag{3.5}$$

where l is choosen to be $0 < l < \frac{2}{n-2}$, such that

$$q + l < \frac{2n-2}{n-2} + l < \frac{2n}{n-2}.$$

Therefore, the embedding $H_0^1(\Omega) \hookrightarrow L^{q+l}(\Omega)$ and $L^p(\Omega) \hookrightarrow L^2(\Omega)$ satisfies

$$\begin{aligned}
\int_\Omega |u|^q \ln|u|\, dx &\le C_{q+l} \|\nabla u\|^{q+l} \\
&= C_{q+l} \|\nabla u\|^2 \|\nabla u\|^{q-2+l} \\
&= C_{q+l} \|\nabla u\|^2 \left(\|\nabla u\|^2\right)^{\frac{q-2+l}{2}} \\
&\le C_{q+l} \left(\frac{2qE(0)}{q-2}\right)^{\frac{q-2+l}{2}} \|\nabla u\|^2,
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\int_\Omega |u|^q \ln|u|\, dx &\le C_{q+l} \|\nabla u\|^{q+l} \\
&= C_{q+l} \|\nabla u\|^p \|\nabla u\|^{q-p+l} \\
&\le C_{q+l} \|\nabla u\|_p^p \|\nabla u\|_p^{q-p+l} \\
&= C_{q+l} \|\nabla u\|_p^p \left(\|\nabla u\|_p^p\right)^{\frac{q-p+l}{p}} \\
&\le C_{q+l} \left(\frac{pq}{q-p} E(0)\right)^{\frac{q-p+l}{p}} \|\nabla u\|_p^p,
\end{aligned} \tag{3.7}$$

here C_{q+l} is the embedding constant.

As a result, by (3.1) and (3.2), we infer that

$$I(t) > \|\nabla u\|^2 + \|\nabla u\|_p^p - \beta\left(\|\nabla u\|^2 + \|\nabla u\|_p^p\right) > 0,\ \forall t \in [0, T^*]. \tag{3.8}$$

By repeating this procedure, T^* can be extended to T. Q.E.D.

Theorem 3.2. If the initial data u_0, u_1 satisfy the conditions of Lemma 3.1, then the solution of (2.2) is uniformly bounded and global in time.

Proof. It suffices to show that $\|\nabla u\|^2 + \|\nabla u\|_p^p + \|u_t\|^2$ is bounded independently of t. We see that,

$$\begin{aligned} E(0) &\geq E(t) = \frac{1}{2}\|u_t\|^2 + J(t) \\ &\geq \frac{1}{2}\|u_t\|^2 + \frac{k}{q^2}\|u\|_q^q \\ &\quad + \frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s(\xi + \mu_2(s)) z^2 ds d\rho dx + \frac{1}{q} I(t) \\ &\geq \frac{1}{2}\|u_t\|^2 + \frac{1}{q}(1-\beta)\left(\|\nabla u\|^2 + \|\nabla u\|_p^p\right). \end{aligned}$$

Thus,

$$\|\nabla u\|^2 + \|\nabla u\|_p^p + \|u_t\|^2 \leq CE(0),$$

where C is a positive constant depending only on k, p and C_{q+1}. Q.E.D.

4 Exponential decay

In this section, we establish the decay results. Firstly, we have the lemmas as follows:

Lemma 4.1. [9] The functional

$$F_1(t) = \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} se^{-\rho s}(\xi + \mu_2(s)) |z(x,\rho,s,t)|^2 ds d\rho dx$$

satisfies, along the solution of (2.2), for some c_1, $c_2 > 0$,

$$F_1'(t) \leq c_1\|u_t\|^2 - c_2\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s(\xi + \mu_2(s)) |z(x,\rho,s,t)|^2 ds d\rho dx. \tag{4.1}$$

Lemma 4.2. The functional

$$F_2(t) = NE(t) + \varepsilon\int_\Omega uu_t dx + \frac{\varepsilon\mu_1}{2}\int_\Omega |u|^2 dx$$

satisfies, along the solution of (2.2)

$$\begin{aligned} F_2'(t) &\leq -(NC_0 - \varepsilon)\|u_t\|^2 - \varepsilon(1-\beta-\delta)\|\nabla u\|^2 \\ &\quad -\varepsilon(1-\beta)\|\nabla u\|_p^p - \left(NC_0 - \varepsilon\frac{c_*}{4\delta}\right)\int_\Omega \int_{\tau_1}^{\tau_2} z^2(x,1,s,t) ds dx, \end{aligned} \tag{4.2}$$

where N, α and ε are positive constants.

Proof. Differentiation, by using equations in (2.2), satisfies

$$\begin{aligned} F_2'(t) &\leq -NC_0\int_\Omega \left(|u_t|^2 + |z(x,1,s,t)|^2\right) dx \\ &\quad +\varepsilon\left(\int_\Omega |u_t|^2 dx - \int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^q \ln|u|^k dx\right) \\ &\quad -\varepsilon\|\nabla u\|_p^p - \varepsilon\int_\Omega u\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t) ds dx. \end{aligned} \tag{4.3}$$

Utilizing Young's inequality and the boundness property of $\mu_2(s)$, we obtain, for any $\delta > 0$ and some $c_* > 0$,

$$\begin{aligned} & -\int_\Omega u \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)\, ds dx \\ \leq\ & \delta \|\nabla u\|^2 + \frac{c_*}{4\delta} \int_\Omega \int_{\tau_1}^{\tau_2} z^2(x,1,s,t)\, ds dx. \end{aligned} \tag{4.4}$$

Q.E.D.

Theorem 4.3. Assume that (3.2) holds. Then, there exist two positive constants c_3 and c_4 such that

$$E(t) \leq c_3 e^{-c_4 t}.$$

Proof. Setting

$$F_3(t) = F_1(t) + F_2(t).$$

It is easy to verify, for ε small enough, that

$$F_3(t) \sim E(t). \tag{4.5}$$

By (4.1) and (4.2), we obtain

$$\begin{aligned} F_3'(t) \leq\ & -(NC_0 - \varepsilon - c_1)\|u_t\|^2 - \varepsilon(1-\beta-\delta)\|\nabla u\|^2 \\ & -\varepsilon(1-\beta)\|\nabla u\|_p^p - \left(NC_0 - \varepsilon\frac{c_*}{4\delta}\right)\int_\Omega \int_{\tau_1}^{\tau_2} z^2(x,1,s,t)\, ds dx \\ & -c_2 \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s(\xi + \mu_2(s))|z(x,\rho,s,t)|^2\, ds d\rho dx. \end{aligned} \tag{4.6}$$

Since $\beta < 1$, choosing δ small enough, such that $\alpha = 1 - \beta - \delta > 0$.

For some $\omega > 0$, the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ satisfies

$$\begin{aligned} \|u\|_q^q \leq\ & C\|\nabla u\|_2^q \\ \leq\ & C\left(\|\nabla u\|^2\right)^{\frac{q-2}{2}} \|\nabla u\|^2 \\ \leq\ & C(E(0))^{\frac{q-2}{2}} \|\nabla u\|^2 \\ \leq\ & \omega\|\nabla u\|^2, \end{aligned}$$

or

$$-\frac{\varepsilon\alpha\omega^{-1}}{2}\|u\|_q^q \geq -\frac{\varepsilon\alpha}{2}\|\nabla u\|_2^2.$$

Hence, (4.6) takes the form

$$\begin{aligned} F_3'(t) \leq\ & -(NC_0 - \varepsilon - c_1)\|u_t\|^2 - \frac{\varepsilon\alpha}{2}\|\nabla u\|^2 - \frac{\varepsilon\alpha\omega^{-1}}{2}\|u\|_q^q \\ & -\varepsilon(1-\beta)\|\nabla u\|_p^p - \left(NC_0 - \varepsilon\frac{c_p}{4\delta}\right)\int_\Omega \int_{\tau_1}^{\tau_2} z^2(x,1,s,t)\, ds dx \\ & -c_2 \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s(\xi + \mu_2(s))|z(x,\rho,s,t)|^2\, ds d\rho dx. \end{aligned} \tag{4.7}$$

Whence δ is fixed, choosing N to be large enough, such that

$$NC_0 - \varepsilon - c_1 > 0,\ NC_0 - \varepsilon \frac{c_p}{4\delta} > 0 \text{ and } 1 - \beta > 0.$$

Therefore, (4.7) takes the form, for some $C > 0$,

$$\begin{aligned} F_3'(t) &\leq -C\left[\|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u\|_p^p + \|u\|_q^q + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s\left(\xi + \mu_2(s)\right) z^2 ds d\rho dx\right] \\ &\leq -CE(t). \end{aligned}$$

By the equivalence relation (4.5) and a simple integration over $(0,t)$, our result proved. Q.E.D.

5 Blow up

In this section, we prove the blow up results for negative initial energy. We have the assumption: $\mu_2 : [\tau_1, \tau_2] \to R$ is an L^∞ function such that:

$$\left(\frac{2\delta - 1}{2}\right) \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \leq \mu_1,\ \delta > \frac{1}{2}. \tag{5.1}$$

Lemma 5.1. Suppose that (5.1) hold. Let u be a solution of (2.2). Then, $\mathcal{K}(t)$ is nonincreasing, such that

$$\begin{aligned} \mathcal{K}(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{p}\|\nabla u\|_p^p + \frac{k}{q^2}\|u\|_q^q \\ &\quad -\frac{1}{q}\int_\Omega |u|^q \ln |u|^k dx + \frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\left|z^2(x,\rho,s,t)\right| ds d\rho dx, \end{aligned} \tag{5.2}$$

which satisfies

$$\mathcal{K}'(t) \leq -c_1\left(\|u_t\|^2 + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)|\left|z^2(x,1,s,t)\right| ds dx\right). \tag{5.3}$$

Proof. By multiplying the first equation of (2.2) by u_t and integrating over Ω , we obtain

$$\begin{aligned} &\frac{d}{dt}\left\{\begin{array}{c} \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{p}\|\nabla u\|_p^p \\ +\frac{k}{q^2}\|u\|_q^q - \frac{1}{q}\int_\Omega |u|^q \ln|u|^k dx \end{array}\right\} \\ &= -\mu_1\|u_t\|^2 - \int_\Omega u_t \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,|z(x,1,s,t)|\, ds dx, \end{aligned} \tag{5.4}$$

and

$$
\begin{aligned}
& \frac{d}{dt}\frac{1}{2}\int_{\Omega}\int_{0}^{1}\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| z^2(x,\rho,s,t)\, dsd\rho dx \\
= & -\frac{1}{2}\int_{\Omega}\int_{0}^{1}\int_{\tau_1}^{\tau_2} 2\left|\mu_2(s)\right| zz_{\rho} dsd\rho dx \\
= & \frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right| z^2(x,0,s,t)\, dsdx \\
& -\frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right| z^2(x,1,s,t)\, dsdx \\
= & \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right| ds\right)\left\|u_t\right\|^2 \\
& -\frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right| z^2(x,1,s,t)\, dsdx. \qquad (5.5)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{d}{dt}\mathcal{K}(t) = & -\mu_1\left\|u_t\right\|^2-\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right| u_t z(x,1,s,t)\, dsdx \\
& +\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right| ds\right)\left\|u_t\right\|^2 \\
& -\frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right| z^2(x,1,s,t)\, dsdx. \qquad (5.6)
\end{aligned}
$$

From (5.4) and (5.5), we get (5.2). By using Young's inequality, (5.1) and (5.6), we obtain (5.3). As a result, the proof is completed. Q.E.D.

To establish our main result, we define

$$
\begin{aligned}
H(t) = & -\mathcal{K}(t)=-\frac{1}{2}\left\|u_t\right\|^2-\frac{1}{2}\left\|\nabla u\right\|^2-\frac{1}{p}\left\|\nabla u\right\|_p^p \\
& -\frac{k}{q^2}\left\|u\right\|_q^q+\frac{1}{q}\int_{\Omega}\left|u\right|^q\ln\left|u\right|^k dx \\
& -\frac{1}{2}\int_{\Omega}\int_{0}^{1}\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right|\left|z^2(x,\rho,s,t)\right| dsd\rho dx. \qquad (5.7)
\end{aligned}
$$

Similar to the work of [10], we have the lemmas as follows:

Lemma 5.2. For $C>0$,

$$
\left(\int_{\Omega}\left|u\right|^q\ln\left|u\right|^k dx\right)^{s/q}\leq C\left[\int_{\Omega}\left|u\right|^q\ln\left|u\right|^k dx+\left\|\nabla u\right\|_2^2+\left\|\nabla u\right\|_p^p\right]
$$

satisfies, for any $u\in L^{q+1}(\Omega)$ and $2\leq s\leq q$, provided that $\int_{\Omega}\left|u\right|^q\ln\left|u\right|^k dx\geq 0$.

Lemma 5.3. Depending on Ω only, suppose that $C>0$, so that

$$\|u\|_2^2 \leq C\left[\left(\int_\Omega |u|^q \ln |u|^k \, dx\right)^{2/q} + \|\nabla u\|_2^{4/q} + \|\nabla u\|_p^{4/p}\right], \tag{5.8}$$

provided that $\int_\Omega |u|^q \ln |u|^k \, dx \geq 0$.

Lemma 5.4. Depending on Ω only, assume that $C>0$, such that

$$\|u\|_q^s \leq C\left[\|u\|_q^q + \|\nabla u\|_2^2 + \|\nabla u\|_p^p\right], \tag{5.9}$$

for any $u \in L^q(\Omega)$ and $2 \leq s \leq q$.

Theorem 5.5. Assume that (5.1) holds. Assume further that

$$\begin{cases} p < q \leq \frac{pn}{n-p}, & \text{if } \; n > p \\ q > p, & \text{if } \; n \leq p, \end{cases}$$

and

$$\mathcal{K}(0) < 0. \tag{5.10}$$

Thus, the solution of (2.2) blows up in finite time.

Proof. By (5.3), we know that

$$\mathcal{K}(t) \leq \mathcal{K}(0) < 0.$$

Thus,

$$\begin{aligned} H'(t) &= -\mathcal{K}'(t) \\ &\geq c_1\left(\|u_t\|^2 + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \, z^2(x,1,s,t) \, ds dx\right) \\ &\geq c_1 \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \, z^2(x,1,s,t) \, ds dx \geq 0 \end{aligned} \tag{5.11}$$

and

$$0 < H(0) \leq H(t) \leq \frac{1}{q} \int_\Omega |u|^q \ln |u|^k \, dx. \tag{5.12}$$

We introduce

$$L(t) = H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t dx + \frac{\mu_1 \varepsilon}{2} \int_\Omega u^2 dx, \; t \geq 0, \tag{5.13}$$

where $\varepsilon > 0$ to be specified later and

$$\frac{2(q-2)}{q^2} < \alpha < \frac{q-2}{2q} < 1 \quad \text{and} \quad 2 < \alpha p q < pq - 4. \tag{5.14}$$

Multiplying the first equation in (2.2) by u and with a derivative of (5.13), we have

$$\begin{aligned} L'(t) &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 \\ &\quad + \varepsilon \int_\Omega u u_{tt} dx + \varepsilon \mu_1 \int_\Omega u u_t dx \\ &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\nabla u\|^2 - \varepsilon \|\nabla u\|_p^p \\ &\quad - \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \, |uz(x,1,s,t)| \, ds dx + \varepsilon \int_\Omega |u|^q \ln |u|^k dx. \end{aligned} \tag{5.15}$$

Thanks to Young's inequality, we get

$$\begin{aligned} & \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \, uz(x,1,s,t) \, ds dx \\ \leq\ & \varepsilon \left[\delta_1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| \, ds \right) \|u\|^2 \right. \\ & \left. + \frac{1}{4\delta_1} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \left| z^2(x,1,s,t) \right| ds dx \right]. \end{aligned} \tag{5.16}$$

Hence, by (5.15), we obtain

$$\begin{aligned} L'(t) &\geq (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\nabla u\|^2 - \varepsilon \|\nabla u\|_p^p \\ &\quad - \varepsilon \delta_1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| \, ds \right) \|u\|^2 - \frac{\varepsilon}{4\delta_1} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \left| z^2(x,1,s,t) \right| ds dx \\ &\quad + \varepsilon \int_\Omega |u|^q \ln |u|^k dx. \end{aligned} \tag{5.17}$$

By using (5.11) and setting δ_1 such that $\frac{1}{4\delta_1 c_1} = \kappa H^{-\alpha}(t)$, we obtain

$$\begin{aligned} L'(t) &\geq [(1-\alpha) - \varepsilon\kappa] H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 \\ &\quad - \varepsilon \|\nabla u\|^2 - \varepsilon \|\nabla u\|_p^p - \varepsilon \frac{H^\alpha(t)}{4c_1\kappa} \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| \, ds \right) \|u\|^2 \\ &\quad + \varepsilon \int_\Omega |u|^q \ln |u|^k dx. \end{aligned} \tag{5.18}$$

For $0 < a < 1$, by (5.18), we have

$$\begin{aligned} L'(t) &\geq [(1-\alpha) - \varepsilon\kappa] H^{-\alpha}(t) H'(t) + \varepsilon a \int_\Omega |u|^q \ln |u|^k dx + \varepsilon \frac{q(1-a)+2}{2} \|u_t\|^2 \\ &\quad + \varepsilon \left(\frac{q(1-a)}{2} - 1 \right) \|\nabla u\|^2 + \varepsilon \left(\frac{q(1-a)}{p} - 1 \right) \|\nabla u\|_p^p \\ &\quad + \frac{\varepsilon(1-a)k}{q} \|u\|_q^q - \varepsilon \frac{H^\alpha(t)}{4c_1\kappa} \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| \, ds \right) \|u\|^2 + \varepsilon q(1-a) H(t) \\ &\quad + \frac{\varepsilon q(1-a)}{2} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| \, |z(x,\rho,s,t)|^2 \, ds d\rho dx. \end{aligned} \tag{5.19}$$

By using (5.8) and (5.12), we get

$$\begin{aligned} H^{\alpha}(t)\|u\|_2^2 &\leq \left(\int_{\Omega}|u|^q \ln|u|^k\,dx\right)^{\alpha}\|u\|_2^2 \\ &\leq \left[\begin{array}{c} \left(\int_{\Omega}|u|^q \ln|u|^k\,dx\right)^{\alpha+2/q} + \left(\int_{\Omega}|u|^q \ln|u|^k\,dx\right)^{\alpha}\|\nabla u\|_2^{4/q} \\ +\left(\int_{\Omega}|u|^q \ln|u|^k\,dx\right)^{\alpha}\|\nabla u\|_p^{4/q} \end{array}\right]. \end{aligned}$$

From Young's inequality, we have

$$\begin{aligned} H^{\alpha}(t)\|u\|_2^2 &\leq \left(\int_{\Omega}|u|^q \ln|u|^k\,dx\right)^{\alpha}\|u\|_2^2 \\ &\leq \left[\begin{array}{c} \left(\int_{\Omega}|u|^q \ln|u|^k\,dx\right)^{(q\alpha+2)/q} \\ +\frac{2}{q}\|\nabla u\|^2 + \frac{q-2}{q}\left(\int_{\Omega}|u|^q \ln|u|^k\,dx\right)^{\alpha q/(q-2)} \\ +\frac{4}{pq}\|\nabla u\|_p^p + \frac{pq-4}{pq}\left(\int_{\Omega}|u|^q \ln|u|^k\,dx\right)^{\alpha pq/(pq-4)} \end{array}\right]. \end{aligned}$$

Hence, we get

$$\begin{aligned} H^{\alpha}(t)\|u\|_2^2 &\leq \left(\int_{\Omega}|u|^q \ln|u|^k\,dx\right)^{\alpha}\|u\|_2^2 \\ &\leq C\left[\begin{array}{c} \left(\int_{\Omega}|u|^q \ln|u|^k\,dx\right)^{(q\alpha+2)/q} + \|\nabla u\|^2 + \|\nabla u\|_p^p \\ +\left(\int_{\Omega}|u|^q \ln|u|^k\,dx\right)^{\alpha q/(q-2)} + \left(\int_{\Omega}|u|^q \ln|u|^k\,dx\right)^{\alpha pq/(pq-4)} \end{array}\right], \end{aligned}$$

where $C = \max\left\{\frac{2}{q},\ \frac{q-2}{q},\ \frac{4}{pq},\ \frac{pq-4}{pq}\right\}$.

By exploiting (5.14), we obtain

$$2 < \alpha q + 2 \leq q,\ 2 < \frac{\alpha q^2}{q-2} \leq q \text{ and } 2 < \alpha pq \leq pq - 4.$$

Thus, lemma 5.2 yields

$$H^{\alpha}(t)\|u\|_2^2 \leq C\left(\int_{\Omega}|u|^q \ln|u|^k\,dx + \|\nabla u\|_2^2 + \|\nabla u\|_p^p\right). \tag{5.20}$$

By combining (5.19) and (5.20), we get

$$\begin{aligned} L'(t) \geq\; & [(1-\alpha)-\varepsilon\kappa] H^{-\alpha}(t) H'(t) \\ & +\varepsilon\left[a-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\right]\int_\Omega |u|^q \ln|u|^k\,dx \\ & +\varepsilon\left[\frac{q(1-a)-2}{2}-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\right]\|\nabla u\|^2 \\ & +\varepsilon\left[\frac{q(1-a)-p}{p}-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\right]\|\nabla u\|_p^p \\ & +\frac{\varepsilon(1-a)k}{q}\|u\|_q^q+\varepsilon\frac{q(1-a)+2}{2}\|u_t\|^2+\varepsilon q(1-a)H(t) \\ & +\frac{\varepsilon q(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\,|z(x,\rho,s,t)|^2\,ds\,d\rho\,dx. \end{aligned} \tag{5.21}$$

Since, choosing $a>0$ so small, such that

$$\frac{q(1-a)-2}{2}>0,$$

and choosing κ large enough, we get

$$\begin{cases} \frac{q(1-a)-2}{2}-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)>0, \\ a-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)>0, \\ \frac{q(1-a)-p}{p}-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)>0. \end{cases}$$

Once κ and a are fixed, picking ε so small, such that

$$(1-\alpha)-\varepsilon\kappa>0,$$

$$H(0)+\varepsilon\int_\Omega u_0u_1\,dx>0.$$

Thus, for some $\lambda>0$, estimate (5.21) takes the form

$$\begin{aligned} L'(t) \geq\; & \lambda\Big[H(t)+\|u_t\|^2+\|\nabla u\|^2+\|\nabla u\|_p^p+\|u\|_q^q \\ & +\int_\Omega |u|^q\ln|u|^k\,dx+\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(\mu_2(s))\,|z(x,\rho,s,t)|^2\,ds\,d\rho\,dx\Big], \end{aligned} \tag{5.22}$$

and

$$L(t)\geq L(0)>0,\ t\geq 0. \tag{5.23}$$

From the embedding $\|u\|_2\leq c\|u\|_q$ and Hölder's inequality, we get

$$\int_\Omega uu_t\,dx\leq \|u\|_2\|u_t\|_2\leq c\|u\|_q\|u_t\|_2,$$

then from Young's inequality, we have

$$\left|\int_{\Omega} uu_t dx\right|^{1/(1-\alpha)} \leq c\left(\|u\|_q^{\mu/(1-\alpha)} + \|u_t\|_2^{\theta/(1-\alpha)}\right), \text{ for } 1/\mu + 1/\theta = 1. \tag{5.24}$$

From Lemma 5.4, we take $\theta = 2(1-\alpha)$ which gives $\mu/(1-\alpha) = 2/(1-2\alpha) \leq q$. Therefore, for $s = 2/(1-2\alpha)$, the estimate (5.24) satisfies

$$\left|\int_{\Omega} uu_t dx\right|^{1/(1-\alpha)} \leq c\left(\|u\|_q^s + \|u_t\|_2^2\right).$$

Therefore, Lemma 5.4 satisfies

$$\left|\int_{\Omega} uu_t dx\right|^{1/(1-\alpha)} \leq c\left[\|\nabla u\|^2 + \|u_t\|^2 + \|u\|_q^q\right]. \tag{5.25}$$

Hence,

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \left(H^{1-\alpha}(t) + \varepsilon\int_{\Omega} uu_t dx + \frac{\mu_1\varepsilon}{2}\int_{\Omega} u^2 dx\right)^{1/(1-\alpha)} \\ &\leq c\left[H(t) + \left|\int_{\Omega} uu_t dx\right|^{1/(1-\alpha)} + \|u\|_2^{2/(1-\alpha)}\right] \\ &\leq c\left[H(t) + \left|\int_{\Omega} uu_t dx\right|^{1/(1-\alpha)} + \|u\|_q^{2/(1-\alpha)}\right] \\ &\leq c\left[H(t) + \|\nabla u\|^2 + \|u_t\|^2 + \|u\|_q^q\right], \ t \geq 0. \end{aligned} \tag{5.26}$$

By combining (5.22) and (5.26), we get

$$L'(t) \geq \Lambda L^{1/(1-\alpha)}(t), \ t \geq 0, \tag{5.27}$$

where Λ is a positive constant depending only on λ and c. A simple integration of (5.27) over $(0,t)$ yields

$$L^{\alpha/(1-\alpha)}(t) \geq \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Lambda\alpha t/(1-\alpha)}.$$

Thus, $L(t)$ blows up in time T^*

$$T \leq T^* = \frac{1-\alpha}{\Lambda\alpha L^{\alpha/(1-\alpha)}(0)}.$$

As a result, the proof is completed. Q.E.D.

6 Conclusions

Recently, there has been published much works concerning the wave equations (Kirchhoff, Petrovsky, Bessel,... etc.) with different state of delay time (constant delay, time-varying delay,... etc.). However, to the best of our knowledge, there were no existence, exponential decay and blow up of solutions for the logarithmic p-Laplacian equation with distributed delay. We have been established the global existence, exponential decay and blow up results for the logarithmic p-Laplacian equation with distributed delay under appropriate conditions.

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